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Eigenvalue Asymptotics for the Schrödinger Operator with a Matrix Potential in a Single Resonance Domain

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Abstract. We consider a Schrödinger Operator with a matrix potential defined in $L_2^m(F)$ by the differential expression

$$L(\phi(x)) = (-\Delta + V(x))\phi(x)$$

and the Neumann boundary condition, where F is the d dimensional rectangle and V is a martix potential, $m \geqslant 2, d \geqslant 2$. We obtain the asymptotic formulas of arbitrary order for the single resonance eigenvalues of the Schrödinger operator in $L_2^m(F)$.

1. Introduction

We consider the Schrödinger Operator with a matrix potential $V(x)$ defined by the differential expression

$$L(\phi(x)) = (-\Delta + V(x))\phi(x) \tag{1}$$

and the Neumann boundary condition

$$\frac{\partial \phi}{\partial n}|_{\partial F} = 0, \tag{2}$$

in $L_2^m(F)$ where F is the d dimensional rectangle $F = [0, a_1] \times [0, a_2] \times \ldots \times [0, a_d]$, $a_1, a_2, \ldots, a_d$ are arbitrary real numbers, ∂F is the boundary of F, $m \geqslant 2, d \geqslant 2$, $\frac{\partial}{\partial n}$ denotes differentiation along the outward normal of the boundary ∂F, $\mathbf{\Delta}$ is a diagonal $m \times m$ matrix whose diagonal elements are the scalar Laplace operators $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_d^2}$, $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, V is a real valued symmetric matrix $V(x) = (v_{ij}(x)), i, j = 1, 2, \ldots, m, v_{ij}(x) \in L_2(F)$, that is, $V^T(x) = V(x)$.

We denote the operator defined by (1)-(2) by $L(V)$, and the eigenvalues and corresponding eigenfunctions of $L(V)$ by Λ_N and Ψ_N, respectively.

The eigenvalues of the operator $L(0)$ which is defined by (1) when $V(x) = 0$ and the boundary condition (2) are $|\gamma|^2$ and the corresponding eigenspaces are

$$E_\gamma = span\{\Phi_{\gamma,1}(x), \Phi_{\gamma,2}(x), \ldots, \Phi_{\gamma,m}(x)\},$$

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where $\gamma \in \frac{\Gamma^{+0}}{2} = \{(\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2} \cdots, \frac{n_d\pi}{a_d}) : \quad n_k \in Z^+ \bigcup\{0\}, \quad k = 1, 2, \ldots, d\}$,

$\Phi_{\gamma,j}(x) = (0, \ldots, 0, u_\gamma(x), 0, \ldots, 0), \quad j = 1, 2, \ldots, m, \quad u_\gamma(x) = cos\frac{n_1\pi}{a_1}x_1 cos\frac{n_2\pi}{a_2}x_2 \cdots cos\frac{n_d\pi}{a_d}x_d$, $u_0(x) = 1$ when $\gamma = (0, 0, \ldots, 0)$. We note that the non-zero component $u_\gamma(x)$ of $\Phi_{\gamma,j}(x)$ stands in the jth component.

It can be easily calculated that the norm of $u_\gamma(x)$, $\gamma = (\gamma^1, \gamma^2, \ldots, \gamma^d) \in \frac{\Gamma^{+0}}{2}$ in $L_2(F)$ is $\sqrt{\frac{\mu(F)}{|A_\gamma|}}$, where $\mu(F)$ is the measure of the d-dimensional parallelepiped F, $| A_\gamma |$ is the number of vectors in

$$A_\gamma = \left\{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \frac{\Gamma}{2} : | \alpha_k | = | \gamma^k |, \ k = 1, 2, \ldots, d\right\},$$

$$\frac{\Gamma}{2} = \left\{\left(\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2} \cdots, \frac{n_d\pi}{a_d}\right) : \ n_k \in Z, \quad k = 1, 2, \ldots, d\right\}.$$

Since $\{u_\gamma(x)\}_{\gamma \in \frac{\Gamma^{+0}}{2}}$ is a complete system in $L_2(F)$, for any $q(x)$ in $L_2(F)$ we have

$$q(x) = \sum_{\gamma \in \frac{\Gamma^{+0}}{2}} \frac{| A_\gamma |}{\mu(F)}(q, u_\gamma)u_\gamma(x), \tag{3}$$

where $(\cdot, \cdot)$ is the inner product in $L_2(F)$.

In our study, it is convenient to use the equivalent decomposition (see [8])

$$q(x) = \sum_{\gamma \in \frac{\Gamma}{2}} q_\gamma u_\gamma(x), \tag{4}$$

where $q_\gamma = \frac{1}{\mu(F)}(q(x), u_\gamma(x))$ for the sake of simplicity. That is, the decomposition (3) and (4) are equivalent for any $d \geq 1$.

Each matrix element $v_{ij}(x) \in L_2(F)$ of the matrix $V(x)$ can be written in its Fourier series expansion

$$v_{ij}(x) = \sum_{\gamma \in \frac{\Gamma}{2}} v_{ij\gamma} u_\gamma(x) \tag{5}$$

for $i, j = 1, 2, \ldots, m$ where $v_{ij\gamma} = \frac{(v_{ij}, u_\gamma)}{\mu(F)}$.

We assume that the Fourier coefficients $v_{ij\gamma}$ of $v_{ij}(x)$ satisfy

$$\sum_{\gamma \in \frac{\Gamma}{2}} | v_{ij\gamma} |^2 (1 + | \gamma |^{2l}) < \infty, \tag{6}$$

for each $i, j = 1, 2, \ldots, m$, $l > \frac{(d+20)(d-1)}{2} + d + 3$ which implies

$$v_{ij}(x) = \sum_{\gamma \in \Gamma^{+0}(\rho^\alpha)} v_{ij\gamma} u_\gamma(x) + O(\rho^{-p\alpha}), \tag{7}$$

where $\Gamma^{+0}(\rho^\alpha) = \{\gamma \in \frac{\Gamma}{2} : 0 \leq | \gamma | < \rho^\alpha\}$, $p = l - d$, $\alpha < \frac{1}{d+20}$, ρ is a large parameter and $O(\rho^{-p\alpha})$ is a function in $L_2(F)$ with norm of order $\rho^{-p\alpha}$. Furthermore, by (6), we have

$$M_{ij} \equiv \sum_{\gamma \in \frac{\Gamma}{2}} | v_{ij\gamma} | < \infty, \text{ for all } \ i, j = 1, 2, \ldots, m. \tag{8}$$

Notice that, if a function $q(x)$ is sufficiently smooth$\Big(q(x) \in W_2^l(F)\Big)$ and the support of $\nabla q(x) = \left(\frac{\partial q}{\partial x_1}, \frac{\partial q}{\partial x_2}, \ldots, \frac{\partial q}{\partial x_d}\right)$ is contained in the interior of the domain F, then $q(x)$ satisfies condition (6) (See [7]). There is also another class of functions $q(x)$, such that $q(x) \in W_2^l(F)$,

$$q(x) = \sum_{\gamma' \in \Gamma} q_{\gamma'} u_{\gamma'}(x),$$

which is periodic with respect to a lattice $\Omega = \{(m_1a_1, m_2a_2, \ldots, m_da_d) : m_k \in \boldsymbol{Z}, k = 1, 2, \ldots, d\}$ and thus it also satisfies condition (6).

One of the essential problems related to this operator $L(V)$ is how the eigenvalues $|\gamma|^2$ of the unperturbed operator $L(0)$ is affected under perturbation. We study this problem by using energy as a large parameter, in other words when $|\gamma| \sim \rho$, that is, there exist positive constants c_1, c_2 such that $c_1\rho < |\gamma| < c_2\rho$, c_1, c_2 do not depend on ρ and ρ is a big parameter. In the sequel, we denote by c_i, $i = 1, 2, \ldots$, the positive constants which does not depend on ρ.

For the scalar case, $m = 1$, a method in which for the first time the eigenvalues of the unperturbed operator $L(0)$ were divided into two groups: non-resonance ones and resonance ones was first introduced by O. Veliev in [16] and more recently in [17], [18] to obtain various asymptotic formulas for the eigenvalues of the periodic Schrödinger operator with quasiperiodic boundary conditions corresponding to each group. By some other methods, asymptotic formulas for quasiperiodic boundary conditions in two and three dimensional cases are obtained in [5], [6], [11], [12] and [7]. When this operator is considered with Dirichlet boundary condition in two dimensional rectangle, the asymptotic formulas for the eigenvalues are obtained in [7]. The asymptotic formulas for the eigenvalues of the Schrödinger operator with Dirichlet or Neumann boundary conditions in an arbitrary dimension are obtained in [1], [8] and [9]. For the matrix case, asymptotic formulas for the eigenvalues of the Schrödinger operator with quasiperiodic boundary conditions are obtained in [12].

As in [16]- [18], we divide $\mathbb{R}^d$ into two domains: Resonance and Non-resonance domains.
In order to define these domains, let us introduce the following sets:

Let $\alpha < \frac{1}{d+20}$, $\alpha_k = 3^k\alpha$, $k = 1, 2, \ldots, d-1$ and

$$V_b(\rho^{\alpha_1}) \equiv \left\{x \in R^d : \quad \left||x|^2 - |x+b|^2\right| < \rho^{\alpha_1}\right\},$$

$$E_1(\rho^{\alpha_1}, p) \equiv \bigcup_{b \in \Gamma(p\rho^\alpha)} V_b(\rho^{\alpha_1}),$$

$$U(\rho^{\alpha_1}, p) \equiv R^d \setminus E_1(\rho^{\alpha_1}, p),$$

$$E_k(\rho^{\alpha_k}, p) = \bigcup_{\gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma(p\rho^\alpha)} \left(\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})\right),$$

where $\Gamma(p\rho^\alpha) \equiv \left\{b \in \frac{\Gamma}{2} : 0 < |\, b \,| < p\rho^\alpha\right\}$ and the intersection $\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})$ in E_k is taken over $\gamma_1, \gamma_2, \ldots, \gamma_k$ which are linearly independent vectors and the length of γ_i is not greater than the length of the other vector in $\Gamma \bigcap \gamma_i\mathbb{R}$. The set $U(\rho^{\alpha_1}, p)$ is said to be a non-resonance domain, and the eigenvalue $|\gamma|^2$ is called a non-resonance eigenvalue if $\gamma \in U(\rho^{\alpha_1}, p)$. The domains $V_b(\rho^{\alpha_1})$, for $b \in \Gamma(p\rho^\alpha)$ are called resonance domains and the eigenvalue $|\gamma|^2$ is a resonance eigenvalue if $\gamma \in V_b(\rho^{\alpha_1})$.

As noted in [17] and [18], the domain $V_b(\rho^{\alpha_1}) \setminus E_2$, called a single resonance domain, has asymptotically full measure on $V_b(\rho^{\alpha_1})$, that is, if

$$2\alpha_2 - \alpha_1 + (d+3)\alpha < 1 \text{ and } \alpha_2 > 2\alpha_1 \tag{9}$$

hold, then

$$\frac{\mu\left(\left(V_b(\rho^{\alpha_1}) \setminus E_2\right) \bigcap B(\rho)\right)}{\mu\left(V_b(\rho^{\alpha_1}) \bigcap B(\rho)\right)} \to 1, \text{ as } \rho \to \infty,$$

where $B(\rho) = \left\{x \in \mathbb{R}^d : |x| \leqslant \rho\right\}$, for a large parameter $\rho >> 1$ and $E_2 = \bigcup_{\gamma_1,\gamma_2 \in \Gamma(p\rho^\alpha)} \left(V_{\gamma_1}(\rho^{\alpha_2}) \bigcap V_{\gamma_2}(\rho^{\alpha_2})\right)$. Since $\alpha < \frac{1}{d+20}$, the conditions in (9) hold.

When $m \geq 2$, in [3], in an arbitrary dimension, the asymptotic formulas of arbitrary order for the eigenvalue of the operator $L(V)$ which corresponds to the non-resonance eigenvalue $|\gamma|^2$ of $L(0)$ are obtained.

In this paper, we obtain the high energy asymptotics of arbitrary order in an arbitrary dimension ($d \geq 2$) for the eigenvalue of $L(V)$ corresponding to resonance eigenvalue $|\gamma|^2$ when γ belongs to the single resonance domain, that is, $\gamma \in V_\delta(\rho^{\alpha_1}) \setminus E_2$, where δ is from $\{e_1, e_2, \ldots, e_d\}$ and $e_1 = \left(\frac{\pi}{a_1}, 0, \ldots, 0\right), \ldots, e_d = \left(0, \ldots, \frac{\pi}{a_d}\right)$.

2. Eigenvalues In a Special Single Resonance Domain

Now let $H_\delta = \{x \in \mathbb{R} : x \cdot \delta = 0\}$ be the hyperplane which is orthogonal to δ. Then we define the following sets:

$$\Omega_\delta = \{\omega \in \Omega : w \cdot \delta = 0\} = \Omega \cap H_\delta,$$

$$\Gamma_\delta = \{\gamma \in \frac{\Gamma}{2} : \gamma \cdot \delta = 0\} = \frac{\Gamma}{2} \cap H_\delta.$$

Here " $\cdot$ " denotes the inner product in $\mathbb{R}^d$. Clearly, for all $\gamma \in \frac{\Gamma}{2}$, we have the following decomposition

$$\gamma = j\delta + \beta, \beta = (\beta^1, \ldots, \beta^d) \in \Gamma_\delta, j \in \boldsymbol{Z}. \tag{10}$$

Note that; if $\gamma = j\delta + \beta \in V_\delta(\rho^{\alpha_1}) \setminus E_2$, then

$$|j| < r_1, \quad r_1 = \rho^{\alpha_1}|\delta|^{-2}+1, \quad |\beta^k| > \frac{1}{3}\rho^{\alpha_1}, \ \forall k : e_k \neq \delta. \tag{11}$$

We write the decomposition (3) of $v_{ij}(x)$ as

$$v_{ij}(x) = \sum_{\gamma' \in \frac{\Gamma}{2}} v_{ij\gamma'} u_{\gamma'}(x) = p_{ij}(s) + \sum_{\gamma \in \frac{\Gamma}{2} \setminus \delta\mathbb{R}} v_{ij\gamma} u_\gamma(x) \tag{12}$$

where

$$p_{ij}(s) = \sum_{n \in \boldsymbol{Z}} p_{ijn} \cos ns, \quad p_{ijn} = v_{ij(n\delta)}, \quad s = x \cdot \delta, \ i, j = 1, 2, \ldots, m. \tag{13}$$

In order to obtain the asymptotic formulas for the single resonance eigenvalues $|\gamma|^2$ ($\gamma \in V_\delta(\rho^{\alpha_1}) \setminus E_2$), we consider the operator $L(V)$ as the perturbation of $L(P(s))$ where $L(P(s))$ is defined by the differential expression

$$Lu = -\Delta u + P(s)u \tag{14}$$

and the Neumann boundary condition

$$\frac{\partial u}{\partial n}|_{\partial F} = 0,$$

$$P(s) = \left(p_{ij}(s)\right), \ i, j = 1, 2, \ldots, m. \tag{15}$$

It can be easily verified by the method of separation of variables that the eigenvalues and the corresponding eigenfunctions of $L(P(s))$, indexed by the pairs $(j, \beta) \in \mathbb{Z} \times \Gamma_\delta$, are $\lambda_{j,\beta} = \lambda_j + |\beta|^2$ and $\chi_{j,\beta}(x) = u_\beta(x) \cdot \varphi_j(s) = \left(u_\beta(x)\varphi_{j1}, u_\beta(x)\varphi_{j2}, \ldots, u_\beta(x)\varphi_{jm}\right)$, respectively, where $\beta \in \Gamma_\delta$, λ_j is the eigenvalue and

$\varphi_j(s) = \left(\varphi_{j,1}(s), \varphi_{j,2}(s), \ldots, \varphi_{j,m}(s)\right)$ is the corresponding eigenfunction of the operator $T(P(s))$ defined by the differential expression

$$T(P(s))Y = -\left|\frac{\pi}{a_i}\right|^2 Y'' + P(s)Y \tag{16}$$

and the boundary condition

$$Y'(0) = Y'(\pi) = 0. \tag{17}$$

The eigenvalues of the operator $T(0)$, defined by (16) when $P(s) = 0$ and the boundary condition (17), are $|n\delta|^2 = |\frac{n\pi}{a_i}|^2$ with the corresponding eigenspace $E_n = span\{C_{n,1}(s), C_{n,2}(s), \ldots, C_{n,m}(s)\}$, where $C_{n,i}(s) = (0, \ldots, \cos ns, \ldots, 0)$, $n \in \mathbb{Z}^+ \cup \{0\}$. It is well known that (For example, see in [15]) the eigenvalue λ_j of $T(P(s))$ satisfying $|\lambda_j - |j\delta|^2| < \sup P(s)$, satisfies the following relation

$$\lambda_j = |j\delta|^2 + O\left(\frac{1}{|j\delta|}\right). \tag{18}$$

By the above equation, the eigenvalue $|\gamma|^2 = |\beta|^2 + |j\delta|^2$ of $L(0)$ corresponds to the eigenvalue $|\beta|^2 + \lambda_j$ of $L(P(s))$.

Note that, we denote the inner product in $L_2^m(F)$ by $\langle \cdot, \cdot \rangle$ which is defined by using the inner product $(\cdot, \cdot)$ in $L_2(F)$ as follows:

$$f(x) = (f_1(x), \ldots, f_m(x)),\ g(x) = (g_1(x), \ldots, g_m(x)) \in L_2^m(F) \ \Rightarrow\ \langle f, g \rangle = (f_1, g_1) + \ldots + (f_m, g_m), \tag{19}$$

for $x \in \mathbb{R}^d$, $d \geq 1$. Also for any $f \in L_2^m[0, \pi]$, since $\{C_{n,i}\}_{n \in \mathbb{Z}^+ \cup \{0\},\ i=1,2,\ldots,m}$ is a complete system, by (19) we have the decomposition

$$\begin{aligned} f(s) &= \sum_{n \in \mathbb{Z}^+ \cup \{0\}} \sum_{i=1}^{m} \frac{2}{\pi} \langle f(s), C_{n,i}(s) \rangle C_{n,i}(s) \\ &= \left(\sum_{n \in \mathbb{Z}^+ \cup \{0\}} \frac{2}{\pi} (f_1(s), \cos ns) \cos ns, \ldots, \sum_{n \in \mathbb{Z}^+ \cup \{0\}} \frac{2}{\pi} (f_m(s), \cos ns) \cos ns \right). \end{aligned} \tag{20}$$

On the other hand, by equivalence of the decompositions (3) and (4) ($q(x) = q(s) \in L_2^m[0, \pi]$, when $d = 1$), it is convenient to use the decomposition

$$f(s) = \sum_{n \in \mathbb{Z}} \sum_{i=1}^{m} \frac{1}{\pi} \langle f(s), C_{n,i}(s) \rangle C_{n,i}(s).$$

In the sequel, for the sake of simplicity, we use the brief notation $\langle f(s), C_{n,i}(s) \rangle$ instead of $\frac{1}{\pi} \langle f(s), C_{n,i}(s) \rangle$, since the constants which do not depend on ρ are inessential in our calculations.

The system of eigenfunctions $\left\{\chi_{j,\beta}\right\}_{j,\beta}$ is complete in $L_2^m(F)$. Indeed; suppose that there exists a non-zero function $f(x) \in L_2^m(F)$ which is orthogonal to each $\chi_{j,\beta}$, $j \in \boldsymbol{Z}$, $\beta \in \Gamma_\delta$. Since $C_{n,i}$, $i = 1, 2, \ldots, m$ can be decomposed by φ_j, by (10), and the definition of $\chi_{j,\beta}$, the function $\Phi_{i,\gamma} = u_\beta(x) \cdot C_{n,i}$, $i = 1, 2, \ldots, m$ can be decomposed by the system $\left\{\chi_{j,\beta}\right\}_{j \in \boldsymbol{Z}, \beta \in \Gamma_\delta}$. Thus, the assumption $\left\langle \chi_{j,\beta}(x)\ ,\ f(x) \right\rangle = 0$ for $j \in \boldsymbol{Z}$, $\beta \in \Gamma_\delta$ implies that $\left\langle f(x)\ ,\ \phi_{i,\gamma} \right\rangle = 0$, $\forall \gamma \in \frac{\Gamma}{2}$ and $i = 1, 2, \ldots, m$, which contradicts to the fact that $\left\{\Phi_{i,\gamma}(x)\right\}_{\gamma \in \frac{\Gamma}{2},\ i=1,\ldots,m}$ is a basis for $L_2^m(F)$.

To prove the asymptotic formulas, we use the binding formula

$$\left(\Lambda_N - \lambda_{j,\beta}\right) \left\langle \psi_N\ ,\ \chi_{j,\beta} \right\rangle = \left\langle \psi_N, (V - P)\chi_{j,\beta} \right\rangle, \tag{21}$$

for the eigenvalue, eigenfunction pairs Λ_N, $\Psi_N(x)$ and $\lambda_{j,\beta}$, $\chi_{j,\beta}$ of the operators $L(V)$ and $L(P(s))$, respectively. The formula (21) can be obtained by multiplying the equation $L(V)\Psi_N(x) = \Lambda_N\Psi_N(x)$ by $\chi_{j,\beta}$ and using the facts that $L(P(s))$ is self-adjoint and $L(P(s))\chi_{j,\beta} = \lambda_{j,\beta}\ \chi_{j,\beta}$.

Now our aim is to decompose $(V-P)\,\chi_{j,\beta}$ with respect to the basis $\left\{\chi_{j',\beta'}\right\}_{j'\in\mathbf{Z},\beta'\in\Gamma_\delta}$. We use the same approach as the one in [9]. This paper contains some additional technicalities:

According to this method, in this paper, it is important to have a suitable Fourier decomposition of vector valued functions to iterate the appropriate binding formulas. For such a decomposition, while studying the decompositions of each component we also need to give a connection with the related binding formulas which is done by (29), Lemma 2.1., (40) and (41). Now the details are as follows:

By (12) and (7), we have

$$v_{ij}(x) - p_{ij}(s) = \sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} d_{ij}(\beta_1,n_1)\cos n_1 s\ u_{\beta_1}(x) + O(\rho^{-p\alpha}), \tag{22}$$

where

$$\Gamma'(\rho^\alpha) = \{(\beta_1,n_1) : \beta_1\in\Gamma_\delta\backslash\{0\}, n_1\in\mathbf{Z}, n_1\delta+\beta_1\in\Gamma(\rho^\alpha)\}$$

and

$$d_{ij}(\beta_1,n_1) = \frac{1}{\mu(F)}\int_F v_{ij}(x)\cos n_1 s\ u_{\beta_1}(x)dx.$$

For $(\beta_1,n_1)\in\Gamma'(p\rho^\alpha)$, we have $|n_1\delta+\beta_1|<p\rho^\alpha$ and since β_1 is orthogonal to δ,

$$|\beta_1|<p\rho^\alpha,\ \ |n_1|<p\rho^\alpha\ \ |n_1|<\frac{1}{2}r_1, \tag{23}$$

(see (11))

Clearly (see equation (22) in [9]), we have, for all $i,j=1,2,\ldots,m$,

$$\sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} d_{ij}\,(\beta_1,n_1)\,(\cos n_1 s)\,u_{\beta_1}(x)u_\beta(x) = \sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} d_{ij}\,(\beta_1,n_1)\,(\cos n_1 s)\,u_{\beta_1+\beta}(x), \tag{24}$$

for all $\beta\in\Gamma_\delta$ satisfying $\left|\beta^k\right|>\frac{1}{3}\rho^{\alpha_1}$, $\forall k : e_k\neq\delta$.

By using the definition of $\chi_{j,\beta}$, $P(s)$, the decompositions (22) and (24), we have

$$\begin{aligned}(V-P)\,\chi_{j,\beta} \quad &= \quad \sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)}\ \sum_{k=1}^{m}\left(d_{1k}\,(\beta_1,n_1)\,(\cos n_1 s)\,\varphi_{j,k}(s)u_{\beta+\beta_1},\ldots,d_{mk}\,(\beta_1,n_1)\,(\cos n_1 s)\,\varphi_{j,k}(s)u_{\beta+\beta_1}\right)\\ &\quad +O\left(\rho^{-p\alpha}\right).\end{aligned} \tag{25}$$

Now we consider the following decompositions:

$$\varphi_{j,k}(s) = \sum_{n\in\mathbf{Z}}\left(\varphi_{j,k},\cos ns\right)\ \cos ns, \tag{26}$$

$$\begin{aligned}\cos n_1 s\ \varphi_{j,k}(s) \quad &= \quad \sum_{n\in\mathbf{Z}}\left(\varphi_{j,k},\cos ns\right).\ \cos n_1 s.\cos ns\\ &= \quad \sum_{n\in\mathbf{Z}}\left(\varphi_{j,k},\cos ns\right).\frac{1}{2}[\cos(n_1+n)s+\cos(n_1-n)s]\\ &= \quad \sum_{n\in\mathbf{Z}}\left(\varphi_{j,k},\cos ns\right).\cos(n_1+n)s,\end{aligned} \tag{27}$$

for each $j \in Z$, $k = 1, 2, \ldots, m$.

On the other hand; the decomposition of $\varphi_j(s) = \left(\varphi_{j,1}(s), \ldots, \varphi_{j,m}(s)\right)$ with respect to the basis $\{C_{n,i}(s) = (0, 0, \ldots, \cos ns, 0, \ldots, 0)\}_{n\in\mathbf{Z}, i=1,2,\ldots,m}$ is given by

$$\begin{aligned}
\varphi_j(s) &= (\varphi_{j,1}, \varphi_{j,2}, \ldots, \varphi_{j,m}) \\
&= \sum_{n\in\mathbf{Z}} \sum_{i=1}^{m} \langle\ \varphi_j(s), C_{n,i}(s)\ \rangle C_{n,i}(s) \\
&= \left(\sum_{n\in\mathbf{Z}} \langle\ \varphi_j(s), C_{n,1}(s)\ \rangle \cos ns, \ldots, \sum_{n\in\mathbf{Z}} \langle\ \varphi_j(s), C_{n,m}(s)\ \rangle \cos ns \right). \qquad (28)
\end{aligned}$$

Thus, (26), (27) and (28), gives

$$\begin{aligned}
\varphi_{j,k}(s) &= \sum_{n\in\mathbf{Z}} \langle\ \varphi_j(s), C_{n,k}(s)\ \rangle\ \cos ns \qquad (29) \\
\cos n_1 s\ \varphi_{j,k}(s) &= \sum_{n\in\mathbf{Z}} \langle\ \varphi_j(s), C_{n,k}(s)\ \rangle\ \cos(n + n_1)s.
\end{aligned}$$

Lemma 2.1. *Let r be a number no less than r_1 ($r \geq r_1$) and j, n be integers satisfying $|j|+1 < r$, $|n| \geq 2r$. Then*

$$\langle\ \varphi_j(s)\ ,\ C_{n,i}(s)\ \rangle = O\left(\rho^{-(l-1)\alpha}\right)\ , \forall i = 1, 2, \ldots, m \qquad (30)$$

and

$$\varphi_j(s) = \sum_{|n|<2r} \sum_{i=1}^{m} \left\langle\ \varphi_j(s)\ ,\ C_{n,i}(s)\ \right\rangle\ C_{n,i}(s) + O\left(\rho^{-(l-2)\alpha}\right). \qquad (31)$$

Proof. We use the following binding formula for $T(0)$ and $T(P(s))$

$$\left(\lambda_j - |n\delta|^2\right)\left\langle \varphi_j(s), C_{n,k}(s)\right\rangle = \langle \varphi_j(s), P(s)C_{n,k}\rangle \qquad (32)$$

and the obvious decomposition, which can be obtained by definition of $P(s)$ and (7),

$$\begin{aligned}
P(s)C_{n,k}(s) &= \left(\sum_{|n_1\delta|<\frac{|n\delta|}{2l}} p_{1kn_1} \cos n_1 s \cos ns, \ldots, \sum_{|n_1\delta|<\frac{|n\delta|}{2l}} p_{mkn_1} \cos n_1 s \cos ns \right) + O\left(|n\delta|^{-(l-1)}\right) \\
&= \left(\sum_{|n_1\delta|<\frac{|n\delta|}{2l}} p_{1kn_1} \cos(n - n_1)s, \ldots, \sum_{|n_1\delta|<\frac{|n\delta|}{2l}} p_{mkn_1} \cos(n - n_1)s \right) + O\left(|n\delta|^{-(l-1)}\right) \\
&= \sum_{t=1}^{m} \sum_{|n_1\delta|<\frac{|n\delta|}{2l}} p_{tkn_1} C_{n-n_1,k}(s) + O\left(|n\delta|^{-(l-1)}\right). \qquad (33)
\end{aligned}$$

Putting above equation (33) into (32), we get

$$\begin{aligned}
\left(\lambda_j - |n\delta|^2\right)\langle \varphi_j(s), C_{n,k}(s)\rangle &= \langle \varphi_j(s), \sum_{t_1=1}^{m} \sum_{|n_1\delta|<\frac{|n\delta|}{2l}} p_{t_1kn_1} C_{n-n_1,k}\rangle + O\left(|n\delta|^{-(l-1)}\right) \\
&= \sum_{t_1=1}^{m} \sum_{|n_1\delta|<\frac{|n\delta|}{2l}} p_{t_1kn_1} \langle \varphi_j(s), C_{n-n_1,k}(s)\rangle + O\left(|n\delta|^{-(l-1)}\right) \qquad (34)
\end{aligned}$$

By assumption $|n|\geq 2r$ and $|j|+1 < r$, thus if $|n_1\delta|< \frac{|n\delta|}{2l}$ then $||(n-n_1)\delta|^2-|j||> \frac{|n|}{5}$ which together with (18) imply $|\lambda_j - |(n-n_1)\delta|^2|> c|n\delta|$. So that in (32) if we substitute $(n-n_1)\delta$ instead of $n\delta$, we get

$$\langle \varphi_j(s), C_{n-n_1,k}(s)\rangle = \frac{\langle \varphi_j(s), P(s)C_{n-n_1,k}\rangle}{\lambda_j - |(n-n_1)\delta|^2} \tag{35}$$

Now using (35) in (34), we get

$$\left(\lambda_j - |n\delta|^2\right)\langle \varphi_j(s), C_{n,k}(s)\rangle = \sum_{t_1=1}^{m}\sum_{|n_1\delta|<\frac{|n\delta|}{2l}} \frac{p_{t_1kn_1}\langle \varphi_j(s), P(s)\ C_{n-n_1,k}(s)\rangle}{\left(\lambda_j - |(n-n_1)\delta|^2\right)} + O\left(|n\delta|^{-(l-1)}\right).$$

Again putting (33) into the last equation, we obtain

$$\begin{aligned}\left(\lambda_j - |n\delta|^2\right)\langle \varphi_j(s), C_{n,k}(s)\rangle &= \sum_{t_1=1}^{m}\sum_{|n_1\delta|<\frac{|n\delta|}{2l}} \frac{p_{t_1kn_1}\langle \varphi_j(s), \sum_{t_2=1}^{m}\sum_{|n_2\delta|<\frac{|n\delta|}{2l}} p_{t_2kn_2}\ C_{n-n_1-n_2,k}(s)\rangle}{\left(\lambda_j - |(n-n_1)\delta|^2\right)} + O\left(|n\delta|^{-(l-1)}\right)\\ &= \sum_{t_1,t_2=1}^{m}\sum_{\substack{|n_1\delta|<\frac{|n\delta|}{2l}\\ |n_2\delta|<\frac{|n\delta|}{2l}}} \frac{p_{t_1kn_1}p_{t_2kn_2}\langle \varphi_j(s), C_{n-n_1-n_2,k}(s)\rangle + O\left(|n\delta|^{-(l-1)}\right)}{\left(\lambda_j - |(n-n_1)\delta|^2\right)}.\end{aligned} \tag{36}$$

In this way, iterating $p_1 = [\frac{l}{2}]$ times and dividing both sides of the obtained equation by $\lambda_j - |n\delta|^2$, we have

$$\langle \varphi_j(s)C_{n,k}(s)\rangle = \sum_{t_1,t_2,\ldots,t_{p_1}=1}^{m}\sum_{\substack{|n_1\delta|<\frac{|n\delta|}{2l}\\ |n_2\delta|<\frac{|n\delta|}{2l}\\ \vdots\\ |n_{p_1}\delta|<\frac{|n\delta|}{2l}}} \frac{p_{t_1kn_1}p_{t_2kn_2}\ldots p_{t_{p_1}kn_{p_1}}\langle \varphi_j, C_{n-n_1-\ldots-n_{p_1},k}\rangle}{\Pi_{s=0}^{p_1-1}\left(\lambda_j - |(n-n_1-\ldots-n_s)\delta|^2\right)} + O(|n\delta|^{-(l-1)}) \tag{37}$$

where the integers $n, n_1, \ldots, n_{p_1}$ satisfy the conditions

$$|n_s|< \frac{|n|}{2l}, \quad s = 1,\ldots,p_1, \quad |j|+1 < \frac{|n|}{2}.$$

These conditions and the assumptions $|n|> 2r$, $|j|+1 < r$ imply that

$$||n-n_1-\ldots-n_s|-|j||> \frac{|n|}{5}, \quad s = 0,1,2,\ldots,p_1.$$

This together with (18), give

$$\frac{1}{|\lambda_j - |(n-n_1-\ldots-n_s)\delta|^2} = \frac{1}{\left||j\delta|^2+O\left(\frac{1}{|j\delta|}\right) - |(n-n_1-\ldots-n_s)\,\delta|^2\right|} = O\left(|n\delta|^{-2}\right) \tag{38}$$

for $s = 0,\ldots,p_1-1$. Hence by (37), (38) and (8), we have

$$\langle \varphi_j(s), C_{n,k}(s)\rangle = O\left(|n\delta|^{-(l-1)}\right).$$

Since $|n\delta|\geq 2r \geq r_1 > 2\rho^{\alpha}$, $O\left(|n\delta|^{-(l-1)}\right) = O(\rho^{-(l-1)\alpha})$ from which we get the proof of (30).

To prove (31), we write the Fourier series of $\varphi_j(s)$ with respect to the basis $\{C_{n,1}(s),\dots,C_{n,m}(s)\}_{n\in\mathbb{Z}}$ as follows:

$$\begin{aligned}\varphi_j(s) &= \sum_{n\in\mathbf{Z}}\langle\varphi_j(s),C_{n,k}(s)\rangle C_{n,k}(s)\\ &= \sum_{|n|<2r}\langle\varphi_j(s),C_{n,k}(s)\rangle C_{n,k}(s)+\sum_{|n|\geqslant 2r}\langle\varphi_j(s),C_{n,k}(s)\rangle C_{n,k}(s),\end{aligned}$$

From which together with (30), we get (31). □

Using the first relation (30) in Lemma 2.1 and (29), we also have

$$\cos n_1 s\ \varphi_{j,k}(s)=\sum_{|n|<2r}\langle\ \varphi_j(s)\ ,\ C_{n,k}(s)\ \rangle\ \cos(n+n_1)s+O\left(\rho^{-(l-2)\alpha}\right). \tag{39}$$

Putting this last relation (39) into (25), we get

$$\begin{aligned}&(V-P)\chi_{j,\beta}=\\ &\sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)}\sum_{|n|<2r}\sum_{k=1}^{m}\Big(d_{1k}(\beta_1,n_1)\langle\ \varphi_j(s)\ ,\ C_{n,k}(s)\ \rangle\ \cos(n+n_1)su_{\beta+\beta_1},\dots,\\ &d_{mk}(\beta_1,n_1)\langle\ \varphi_j(s)\ ,\ C_{n,k}(s)\ \rangle\ \cos(n+n_1)su_{\beta+\beta_1}\Big)+O\left(\rho^{-p\alpha}\right).\end{aligned} \tag{40}$$

$\Big($ note that $p=(l-d), d\geq 2\Rightarrow \frac{1}{\rho(l-2)}<\frac{1}{\rho^{p\alpha}}$. Hence $O\left(\rho^{-p\alpha}\right)+O\left(\rho^{-(l-2)\alpha}\right)=O\left(\rho^{-p\alpha}\right).\Big)$

Now, in order to decompose $(V-P)\chi_{j,\beta}$ with respect to $\left\{\chi_{j+j_1',\beta_1'}\right\}$ we consider the inner product $\langle(V-P)\chi_{j,\beta},\chi_{j+j_1',\beta_1'}\rangle$, that is, by the definition of $\chi_{j+j_1',\beta_1'}$ and (40), the inner products $(\cos(n+n_1)s\ u_{\beta+\beta_1}\ ,\ \varphi_{j+j_1',t}(s)\ u_{\beta_1'})$, $t=1,2,\dots,m$. Using the decomposition (29), instead of j, we substitute $j+j_1'$ to get

$$\begin{aligned}\left(\cos(n+n_1)s\ u_{\beta+\beta_1}\ ,\ \varphi_{j+j_1',t}(s)\ u_{\beta_1'}\right) &= \left(\cos(n+n_1)s\ u_{\beta+\beta_1}\ ,\ \sum_{n'\in\mathbf{Z}}\langle\ \varphi_{j+j_1'}(s)\ ,\ C_{n',t}(s)\ \rangle\ \cos n's\ u_{\beta_1'}\right)\\ &= \sum_{n'\in\mathbf{Z}}\langle\ \overline{\varphi_{j+j_1'}(s)\ ,\ C_{n',t}(s)}\ \rangle\left(\cos(n+n_1)s\ u_{\beta+\beta_1},\cos n's\ u_{\beta_1'}\right).\end{aligned}$$

Note that if $\beta_1'\neq\beta+\beta_1$ or $n'\neq n+n_1$ then $(\cos(n+n_1)s\ u_{\beta+\beta_1}\ ,\ \cos n's\ u_{\beta_1'})=0$. Thus,

$$\left(\cos(n+n_1)s\ u_{\beta+\beta_1}\ ,\ \varphi_{j+j_1',t}(s)\ u_{\beta_1'}\right)=\begin{cases}0 & ,\ \text{if}\quad \beta_1'\neq\beta+\beta_1\quad\text{or}\quad n'\neq n+n_1\\ \langle\ \overline{\varphi_{j+j_1'}(s)\ ,\ C_{n+n_1,t}(s)}\ \rangle & ,\ \text{otherwise.}\end{cases}$$

Using the last equality and (40), we get

$$(V-P)\chi_{j,\beta}=\sum_{\substack{j_1'\in\mathbf{Z}\\(\beta_1,n_1)\in\Gamma'(\rho^\alpha)}}\left(\sum_{|n|<2r}\sum_{k=1}^{m}\sum_{i=1}^{m}d_{ik}(n_1,\beta_1)\langle\varphi_j,C_{n,k}\rangle\langle\overline{\varphi_{j+j_1'},C_{n+n_1,i}}\rangle\right)\chi_{j+j_1',\beta+\beta_1}+O(\rho^{-p\alpha}). \tag{41}$$

Lemma 2.2. *Let r be a number no less than r_1 ($r\geq r_1$), j,n and $n_1$be integers satisfying $|n|<2r$, $|n_1|<\frac{1}{2}r_1$ and $|j|+1<r$, then*

$$\sum_{\substack{j_1\in\mathbf{Z}\\|j_1|\geqslant 6r}}\langle\varphi_{j+j_1},C_{n,i}\rangle=O\left(\rho^{-(l-2)\alpha}\right),\forall i=1,2,\dots,m.$$

Proof. By the binding formula (32) for $T(0)$ and $T(P(s))$ we have

$$\left(\lambda_{j+j_1} - |(n+n_1)\delta|^2\right)\langle\varphi_{j+j_1}, C_{n+n_1,k}\rangle = \langle\varphi_{j+j_1}, P(s)C_{n+n_1,k}\rangle. \tag{42}$$

If $|j_1|\geq 6r$ then the assumptions of this lemma imply $||j+j_1|-|n+n_1||> \frac{r}{2}$. Thus, using (42) and the fact that $\lambda_{j+j_1} = |(j+j_1)\delta|^2+O\left(\frac{1}{|(j+j_1)\delta|}\right)$, we get

$$|\sum_{j_1\geqslant 6r}\langle\varphi_{j+j_1}, C_{n+n_1,k}\rangle| = |\sum_{j_1\geqslant 6r}\frac{\langle\varphi_{j+j_1}, P(s)C_{n+n_1,k}\rangle}{\lambda_{j+j_1}-|(n+n_1)\delta|^2}|.$$

Using the decomposition of $p_{tk}(s) = \left(\sum_{|l_1\delta|<|r\delta|} v_{tk,l_1\delta}\cos l_1 s\right)+O(|r\delta|^{-(l-1)})$ and iterating the obtained formula $p_1 = [\frac{l}{2}]$ times as in the proof of Lemma 2.1, we get

$$|\sum_{|j_1|\geqslant 6r}\langle\varphi_{j+j_1}, C_{n+n_1,k}\rangle| = |\sum_{j_1\geqslant 6r}\sum_{\substack{|l_1\delta|<|r\delta|\\ |l_2\delta|<|r\delta|\\ \vdots\\ |l_p\delta|<|r\delta|}}\sum_{t_1,t_2,\ldots,t_p=1}^{m}\frac{v_{t_1k,l_1\delta}\ldots v_{t_pk,l_p\delta}\langle\varphi_{j'}, C_{n+n_1-l_1-\ldots-l_k}\rangle}{\prod_{s=0}^{p-1}|\lambda_{j+j_1}-(n+n_1-l_1-\ldots-l_s)\delta|^2} \tag{43}$$

Since $|n|< 2r$ and $|n_1|< \frac{1}{2}r_1 < \frac{1}{2}r$, $|n+n_1|< \frac{5r}{2}$. Also, $|n+n_1-l_1-\ldots-l_s|< 3r$ and $\frac{1}{|\lambda_{j+j_1}-|(n+n_1-l_1-\ldots-l_s)\delta|^2|} = O\left(|r|^{-2}\right)$. Substituting this result into (43) and using (8), we get the proof. □

By Lemma 2.2, the equation (41) becomes;

$$\begin{aligned}(V-P)\chi_{j,\beta} &= O(\rho^{-p\alpha}) + \sum_{\substack{|j_1'|<6r\\ (\beta_1,n_1)\in\Gamma'(\rho^\alpha)}}\left(\sum_{|n|<2r}\sum_{k=1}^{m}\sum_{i=1}^{m} d_{ik}(n_1,\beta_1)\langle\varphi_j, C_{n,k}\rangle\overline{\langle\varphi_{j+j_1'}, C_{n+n_1,i}\rangle}\right)\chi_{j+j_1',\beta+\beta_1}\\ &= O(\rho^{-p\alpha}) + \sum_{\substack{|j_1|<6r\\ (\beta_1,n_1)\in\Gamma'(\rho^\alpha)}}\left(\sum_{|n|<2r}\sum_{k=1}^{m}\sum_{i=1}^{m} d_{ik}(n_1,\beta_1)\langle\varphi_j, C_{n,k}\rangle\overline{\langle\varphi_{j+j_1}, C_{n+n_1,i}\rangle}\right)\chi_{j+j_1,\beta+\beta_1},\end{aligned}$$

that is,

$$(V-P)\chi_{j,\beta} = \sum_{(\beta_1,j_1)\in\mathbf{Q}(\rho^\alpha,6r)} A(j,\beta,j+j_1,\beta+\beta_1)\,\chi_{j+j_1,\beta+\beta_1} + O(\rho^{-p\alpha}), \tag{44}$$

for every j satisfying $|j|+1 < r$, where

$$\mathbf{Q}(\rho^\alpha,6r) = \{(j,\beta) : |j\delta|< 6r\ ,\ 0<|\beta|<\rho^\alpha\},$$

$$A(j,\beta,j+j_1,\beta+\beta_1) = \sum_{n_1:(n_1,\beta_1)\in\Gamma'(\rho^\alpha)}\left(\sum_{|n|<2r}\sum_{k=1}^{m}\sum_{i=1}^{m} d_{ik}(n_1,\beta_1)\langle\varphi_j, C_{n,k}\rangle\overline{\langle\varphi_{j+j_1}, C_{n+n_1,i}\rangle}\right).$$

We need to prove that

$$\sum_{(\beta_1,j_1)\in\mathbf{Q}(\rho^\alpha,6r)}\left|A(j,\beta,j+j_1,\beta+\beta_1)\right| < c_3. \tag{45}$$

By the definition of $A(j,\beta,j+j_1,\beta+\beta_1)$, $d_{ik}(n_1,\beta_1)$ and (8), we have

$$\sum_{(\beta_1,j_1)\in\mathbf{Q}(\rho^\alpha,6r)}\left|A\left(j,\beta_1,j^{'},\beta+\beta_1\right)\right| \leq \sum_{n_1:(n_1,\beta_1)\in\Gamma^{'}(\rho^\alpha)}\sum_{i,k=1}^{m}\left|d_{ik}(n_1,\beta_1)\right|\sum_{|n|<2r}\left|\langle\varphi_j,C_{n,k}\rangle\right|\sum_{|j_1|<6r}\left|\langle\overline{\varphi_{j+j_1},C_{n+n_1,i}}\rangle\right|$$
$$\leq c_4\sum_{|n|<2r}\left|\langle\varphi_j,C_{n,k}\rangle\right|\sum_{|j_1|<6r}\left|\langle\varphi_{j+j_1},C_{n+n_1,i}\rangle\right| \tag{46}$$

Now we prove that

$$\sum_{n\in\mathbb{Z}}\left|\langle\varphi_j,C_{n,k}\rangle\right|<c_5 \quad \text{and} \quad \sum_{j_1\in\mathbb{Z}}\left|\langle\varphi_{j+j_1},C_{n+n_1,i}\rangle\right|<c_6 \tag{47}$$

For this, let

$$A=\left\{n\in\mathbb{Z}\ :\ |n\delta|^2\in[\lambda_{j-1},\lambda_{j+1}]\ \right\}$$

and

$$B=\left\{j_1\in\mathbb{Z}\ :\ \lambda_{j+j_1}\in\left[\ |(n+n_1)\delta|^2-1,\ \ |(n+n_1)\delta\ |^2+1\ \right]\ \right\},$$

then it follows from (18) that the number of elements in the sets A and B are less than c_7. So if we isolate the terms with $n\in A$ and $j_1\in B$ in the first and second summations of inequalities in (47), respectively, appliying (32) to the other tems then using the facts

$$\sum_{n\notin A}\frac{1}{|\lambda_j-|n\delta|^2|}<c_8, \quad \sum_{j_1\notin\beta}\frac{1}{|\lambda_{j+j_1}-|(n+n_1)\delta|^2|}<c_9$$

we get (47), hence by (46), (45) is proved.

The expressions (44) and (21) together imply that

$$\left(\Lambda_N-\lambda_{j^{'},\beta^{'}}\right)\langle\psi_N\ ,\ \chi_{j^{'},\beta^{'}}\rangle=\sum_{(\beta_1,j_1)\in\mathbf{Q}(\rho^\alpha 6r)}A\left(j^{'},\beta^{'},j^{'}+j_1,\beta^{'}+\beta_1\right)\langle\psi_N\ ,\ \chi_{j^{'}+j_1,\beta^{'}+\beta_1}\rangle+O(\rho^{-p\alpha}). \tag{48}$$

If the condition (iterability condition for the triple $(N,j^{'},\beta^{'})$)

$$|\Lambda_N-\lambda_{j^{'},\beta^{'}}|>c_{10} \tag{49}$$

holds then the formula (48) can be written in the following form

$$\langle\psi_N\ ,\ \chi_{j^{'},\beta^{'}}\rangle=\sum_{(\beta_1,j_1)\in\mathbf{Q}(\rho^\alpha 6r)}\frac{A\left(j^{'},\beta^{'},j^{'}+j_1,\beta^{'}+\beta_1\right)\langle\psi_N\ ,\ \chi_{j^{'}+j_1,\beta^{'}+\beta_1}\rangle}{\Lambda_N-\lambda_{j^{'},\beta^{'}}}+O(\rho^{-p\alpha}). \tag{50}$$

Using (48) and (50), we are going to find Λ_N which is close to $\lambda_{j,\beta}$, where $|j|+1<r_1$.For this, first in (48) instead of $j^{'},\beta^{'}$, taking j,β, hence instead of r taking r_1, we get

$$\left(\Lambda_N-\lambda_{j,\beta}\right)\langle\psi_N\ ,\ \chi_{j,\beta}\rangle=\sum_{(\beta_1,j_1)\in\mathbf{Q}(\rho^\alpha,6r_1)}A\left(j,\beta,j+j_1,\beta+\beta_1\right)\langle\psi_N\ ,\ \chi_{j+j_1,\beta+\beta_1}\rangle+O(\rho^{-p\alpha}). \tag{51}$$

To iterate it by using (50) for $j^{'}=j+j_1$ and $\beta^{'}=\beta+\beta_1$, we will prove that there is a number N such that

$$|\Lambda_N-\lambda_{j+j_1,\beta+\beta_1}|>\frac{1}{2}\rho^{\alpha_2}, \tag{52}$$

where $|j + j_1| < 7r_1 \equiv r_2$, since $\lambda_{j,\beta}$ and $|j_1| < 6r_1$. Then $(j + j_1, \beta + \beta_1)$ satisfies (49). This means that, in formula (48), the pair (j', β') can be replaced by the pair $(j + j_1, \beta + \beta_1)$. Then, (48) instead of r taking r_2, we get

$$\langle \psi_N \, , \, \chi_{j+j_1,\beta+\beta_1} \rangle =$$
$$O(\rho^{-p\alpha}) + \sum_{(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)} \frac{A\,(j + j_1, \beta + \beta_1, j + j_1 + j_2, \beta + \beta_1 + \beta_2)\,\langle \psi_N \, , \, \chi_{j+j_1+j_2,\beta+\beta_1+\beta_2} \rangle}{\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}}.$$

Putting the above formula into (51), we obtain

$$\left(\Lambda_N - \lambda_{j,\beta}\right) c(N, j, \beta) = O(\rho^{-p\alpha}) + \sum_{\substack{(\beta_1,j_1)\in F(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in F(\rho^\alpha,6r_2)}} \frac{A\left(j, \beta, j^1, \beta^1\right) A\left(j^1, \beta^1, j^2, \beta^2\right) c(N, j^2, \beta^2)}{\Lambda_N - \lambda_{j^1,\beta^1}} \tag{53}$$

where $c(N, j, \beta) = \langle \psi_N, \chi_{j,\beta} \rangle$, $j^k = j + j_1 + j_2 + \ldots + j^k$ and $\beta^k = \beta + \beta_1 + \beta_2 + \ldots + \beta_k$. Thus, we are going to find a number N such that $c(N, j, \beta)$ is not too small and the condition (52) is satisfied.

Lemma 2.3. *(a) Suppose $h_1(x), h_2(x), \ldots, h_{p_2}(x) \in L_2^m(F)$ where $p_2 = [\frac{d}{2\alpha_2}] + 1$. Then for every eigenvalue $\lambda_{j,\beta}$ of the operator $L(P(s))$, there exists an eigenvalue Λ_N of $L(V)$ satisfying*

(i) $|\Lambda_N - \lambda_{j,\beta}| < 2M$, *where* $M = \|V\|$,

(ii) $|c(N, j, \beta)| > \rho^{-q\alpha}$, *where* $q\alpha = [\frac{d}{2\alpha} + 2]\alpha$,

(iii) $|c(N, j, \beta)|^2 > \frac{1}{2p_2} \sum_{i=1}^{p_2} |\langle \psi_N, \frac{h_i}{\|h_i\|} \rangle|^2 > \frac{1}{2p_2} |\langle \psi_N, \frac{h_i}{\|h_i\|} \rangle|^2$, $\forall i = 1, 2, \ldots, p_2$.

(b) Let $\gamma = \beta + j\delta \in V_\delta'(\alpha)$ and $(\beta_1, j_1) \in Q(\rho^\alpha, 6r_1), (\beta_k, j_k) \in Q(\rho^\alpha, 6r_k)$, where $r_k = 7r_{k-1}$ for $k = 2, 3, \ldots, p$. Then for $k = 1, 2, 3, \ldots, p$, we have

$$|\lambda_{j,\beta} - \lambda_{j^k,\beta^k}| > \frac{3}{5}\rho^{\alpha_2}, \quad \forall \beta^k \neq \beta. \tag{54}$$

Proof. (a) Let A, B, C be the set of indexes N satisfying (i), (ii), (iii), respectively. Using the binding formula (21) for $L(V)$ and $L(P(s))$ and the Bessel's inequality, we get

$$\sum_{N\notin A} |c(N, j, \beta)|^2 = \sum_{N\notin A} \left| \frac{(\psi_N, (V - P)\chi_{j,\beta})}{\Lambda_N - \lambda_{j,\beta}} \right|^2$$
$$\leq \frac{1}{4M^2} \|(V - P)\chi_{j,\beta}\|^2 \leq \frac{1}{4}.$$

Hence by Parseval's relation, we obtain

$$\sum_{N\in A} |c(N, j, \beta)|^2 > \frac{3}{4}.$$

Using the fact that the number of indexes N in A is less than $\rho^{d\alpha}$ and by the relation $N \notin B \Rightarrow |c(N, j, \beta)| < \rho^{-q\alpha}$, we have

$$\sum_{N\in A\setminus B} |c(N, j, \beta)|^2 < \rho^{d\alpha}\rho^{-q\alpha} < \rho^{-\alpha},$$

since $\alpha < \frac{1}{d+20}$. On the other hand by the relation $A = (A \setminus B) \bigcup (A \bigcap B)$ and the above inequalities, we get

$$\frac{3}{4} < \sum_{N\in A} |c(N, j, \beta)|^2 = \sum_{N\in A\setminus B} |c(N, j, \beta)|^2 + \sum_{N\in A\bigcap B} |c(N, j, \beta)|^2,$$

which implies

$$\sum_{N\in A\bigcap B}|c(N,j,\beta)|^2>\frac{3}{4}-\rho^{-\alpha}>\frac{1}{2}. \tag{55}$$

Now, suppose that $A\bigcap B\bigcap C=\emptyset$, i.e., for all $N\in A\bigcap B$, the condition (iii) does not hold. Then by (55) and Bessel's inequality, we have

$$\begin{aligned}\frac{1}{2}<\sum_{N\in A\bigcap B}|c(N,j,\beta)|^2&\leq\sum_{N\in A\bigcap B}\frac{1}{2p_2}\sum_{i=1}^{p_2}\left|\left\langle\psi_N,\frac{h_i}{\|h_i\|}\right\rangle\right|^2=\frac{1}{2p_2}\sum_{i=1}^{m}\sum_{N\in A\bigcap B}\left|\left\langle\psi_N,\frac{h_i}{\|h_i\|}\right\rangle\right|^2\\&<\frac{1}{2p_2}\sum_{i=1}^{p_2}\left\|\frac{h_i}{\|h_i\|}\right\|^2=\frac{1}{2},\end{aligned}$$

which is a contradiction.

(b) The definition of $\lambda_{j,\beta}$ gives

$$\begin{aligned}|\lambda_{j,\beta}-\lambda_{j^k,\beta^k}|&=\||\beta|^2+\lambda_j-|\beta+\beta_1+...+\beta_k|^2-\lambda_{j^k}|\\&\geq\||\beta|^2-|\beta+\beta_1+...+\beta_k|^2|-|\lambda_j-\lambda_{j^k}\|.\end{aligned} \tag{56}$$

The condition of the lemma $(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)$, $(\beta_k,j_k)\in Q(\rho^\alpha,6r_k)$ and the relation $\beta+j\delta\in V_\delta(\rho^{\alpha_1})\setminus E_2$ together with $|j\delta|<c_{11}\rho^{\alpha_1}$ (see (11)) and $|j_i\delta|<c_{12}\rho^{\alpha_1}$ (see (23)) imply that

$$\begin{aligned}\rho^{\alpha_2}&<\||\beta|^2+|j\delta|^2-|\beta^k|^2-|j^k\delta|^2|\\&<\||\beta|^2-|\beta^k|^2|+c_9\rho^{\alpha_1},\qquad\beta_1+...+\beta_k\neq0,\end{aligned}$$

since $\beta,\beta_1,...,\beta_k$ are orthogonal to δ. That is, we have

$$||\beta|^2-|\beta_k|^2|>c_{13}\rho^{\alpha_2}.$$

This last inequality together with (56) and the asymptotic formula (18) give

$$|\lambda_{j,\beta}-\lambda_{j^k,\beta^k}|>c_{14}\rho^{\alpha_2}.$$

□

3. Asymptotic Formulas

Now we consider the following function

$$h_i(x)=\sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)}}\frac{A\left(j,\beta,j^1,\beta^1\right)A\left(j^1,\beta^1,j^2,\beta^2\right)\chi_{j^{(2)},\beta^{(2)}}}{\left(\lambda_{j,\beta}-\lambda_{j^1,\beta^1}\right)^i},\quad 1\leq i\leq p_2. \tag{57}$$

Since $\left\{\chi_{j^{(2)},\beta^{(2)}}(x)\right\}$ is a total system and $\beta_1\neq0$ by (45) and (54), we have

$$\begin{aligned}\sum_{(j',\beta')}|\langle h_i(x),\chi_{j',\beta'}\rangle|^2&=\sum_{\substack{(\beta_1,j_1)\in F(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in F(\rho^\alpha,6r_2)}}\frac{|A\left(j,\beta,j^1,\beta^1\right)A\left(j^1,\beta^1,j^2,\beta^2\right)|^2}{|\left(\lambda_{j,\beta}-\lambda_{j^1,\beta^1}\right)^i|^2}\\&\leq c_{12}\ \rho^{-2i\alpha_2},\end{aligned} \tag{58}$$

i.e., $h_i(x)\in L_2^m(F)$ and $\|h_i(x)\|=O(\rho^{-i\alpha_2})$, $\forall i=1,2,\ldots,p_2$.

Theorem 3.1. *For every eigenvalue $\lambda_{j,\beta}$ of the operator $L(P(s))$ with $\beta + j\delta \in V_{\delta}^{'}(\rho^{\alpha_1})$, there exists an eigenvalue Λ_N of the operator $L(V)$ satisfying*

$$\Lambda_N = \lambda_{j,\beta} + O\left(\rho^{-\alpha_2}\right). \tag{59}$$

Proof. By Lemma 2.3, for the chosen $h_i(x), i = 1, 2, \ldots, p_2$ in (57), there exists a number N, satisfying (i), (ii), (iii). Since $(\beta_1, j_1) \in Q(\rho^{\alpha}, 6r_1)$, by part (b) of Lemma 2.3, we have

$$|\lambda_{j,\beta} - \lambda_{j^1,\beta^1}| > c_{15}\rho^{\alpha_2}.$$

The above inequality together with (i) imply

$$|\Lambda_N - \lambda_{j^1,\beta^1}| > c_{16}\rho^{\alpha_2}.$$

Using the following well known decomposition

$$\frac{1}{\left[\Lambda_N - \lambda_{j^1,\beta^1}\right]} = \sum_{i=1}^{p_2} \frac{\left[\Lambda_N - \lambda_{j,\beta}\right]^{i-1}}{\left[\lambda_{j,\beta} - \lambda_{j^1,\beta^1}\right]^{i}} + O\left(\rho^{-(p_2+1)\alpha_2}\right),$$

and (57), we see that the formula (53) can be written as

$$\begin{aligned}\left(\Lambda_N - \lambda_{j,\beta}\right) c(N, j, \beta) &= O(\rho^{-p\alpha}) + \sum_{\substack{(\beta_1,j_1)\in F(\rho^{\alpha},6r_1)\\(\beta_2,j_2)\in F(\rho^{\alpha},6r_2)}} \frac{A\left(j, \beta, j^1, \beta^1\right) A\left(j^1, \beta^1, j^2, \beta^2\right) \langle \psi_N, \chi_{j^2,\beta^2} \rangle}{\Lambda_N - \lambda_{j^1,\beta^1}} \\ &= \sum_{i=1}^{p_2} \left[\left(\Lambda_N - \lambda_{j,\beta}\right)^{i-1} \left\langle \psi_N, \frac{h_i}{\|h_i\|} \right\rangle \right] \|h_i\| + O\left(\rho^{-(p_2+1)\alpha_2}\right).\end{aligned}$$

Now dividing both sides of the last equation by $c(n, j, \beta)$ and using (ii), (iii), we have

$$\begin{aligned}&|\Lambda_N - \lambda_{j,\beta}| \leq O\left(\rho^{-(p_2+1)\alpha_2+q\alpha}\right) + \\ &\frac{\left|\left\langle \psi_N, \frac{h_1}{\|h_1\|} \right\rangle\right|}{|c(N, j, \beta)|} \|h_1\| + \frac{|\Lambda_N - \lambda_{j,\beta}| \left|\left\langle \psi_N, \frac{h_2}{\|h_2\|} \right\rangle\right|}{|c(N, j, \beta)|} \|h_2\| + \ldots + \frac{|\Lambda_N - \lambda_{j,\beta}|^{(p_2-1)} \left|\left\langle \psi_N, \frac{h_{p_2}}{\|h_{p_2}\|} \right\rangle\right|}{|c(N, j, \beta)|} \|h_{p_2}\| \\ &\leq (2p_2)^{\frac{1}{2}} \left(\|h_1\| + 2M\|h_2\| + \ldots + (2M)^{p_2-1} \|h_{p_2}\| \right) + O\left(\rho^{-(p_2+1)\alpha_2+q\alpha}\right).\end{aligned}$$

Hence by (58), we obtain

$$\Lambda_N = \lambda_{j,\beta} + O\left(\rho^{-\alpha_2}\right),$$

since $(p_2 + 1)\alpha_2 - q\alpha > \alpha_2$. Theorem is proved. □

It follows from (54) and (59) that the triples (N, j^k, β^k) for $k = 1, 2, \ldots, p_1$, satisfy the iterability condition (49). By (50) instead of $j^{'}, \beta^{'}$ and r taking j^2, β^2 and r_3, we have

$$c(N, j^2, \beta^2) = \sum_{(\beta_3,j_3)\in Q(\rho^{\alpha},6r_3)} \frac{A(j^2, \beta^2, j^3, \beta^3)(\psi_N, \chi_{j^3,\beta^3})}{\Lambda_N - \lambda_{j^2,\beta^2}} + O(\rho^{-p\alpha}). \tag{60}$$

To obtain the other terms of the asymptotic formula of Λ_N, we iterate the formula (53). Now we isolate the terms with multiplicand $c(N, j, \beta)$ in the right hand side of (53).

$$(\Lambda_N - \lambda_{j,\beta})c(N,j,\beta) = O(\rho^{-p\alpha}) + \sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(j+j_1+j_2,\beta+\beta_1+\beta_2)=(j,\beta)}} \frac{A\left(j,\beta,j^1,\beta^1\right)A\left(j^1,\beta^1,j,\beta\right)}{\Lambda_N - \lambda_{j^1,\beta^1}}c(N,j,\beta)$$
$$+ \sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(j+j_1+j_2,\beta+\beta_1+\beta_2)\neq(j,\beta)}} \frac{A\left(j,\beta,j^1,\beta^1\right)A\left(j^1,\beta^1,j^2,\beta^2\right)}{\Lambda_N - \lambda_{j^1,\beta^1}}c(N,j^2,\beta^2). \tag{61}$$

Substituting the equation (60) into the second sum of the equation (61), we get

$$(\Lambda_N - \lambda_{j,\beta})c(N,j,\beta) = \sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(j^2,\beta^2)=(j,\beta)}} \frac{A\left(j,\beta,j^1,\beta^1\right)A\left(j^1,\beta^1,j,\beta\right)}{\Lambda_N - \lambda_{j^1,\beta^1}}c(N,j,\beta)$$
$$+ \sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(j^2,\beta^2)\neq(j,\beta)\\(\beta_3,j_3)\in Q(\rho^\alpha,6r_3)}} \frac{A\left(j,\beta,j^1,\beta^1\right)A\left(j^1,\beta^1,j^2,\beta^2\right)A\left(j^2,\beta^2,j^3,\beta^3\right)}{(\Lambda_N - \lambda_{j^1,\beta^1})(\Lambda_N - \lambda_{j^2,\beta^2})}c(N,j^3,\beta^3)$$
$$+ O(\rho^{-p\alpha}). \tag{62}$$

Again isolating terms $c(N, j, \beta)$ in the last sum of the equation (62), we obtain

$$(\Lambda_N - \lambda_{j,\beta})c(N,j,\beta) = [\sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(j^2,\beta^2)=(j,\beta)}} \frac{A\left(j,\beta,j^1,\beta^1\right)A\left(j^1,\beta^1,j,\beta\right)}{\Lambda_N - \lambda_{j^1,\beta^1}}$$
$$+ \sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(\beta_3,j_3)\in Q(\rho^\alpha,6r_3)\\(j^2,\beta^2)\neq(j,\beta)\\(j^3,\beta^3)=(j,\beta)}} \frac{A\left(j,\beta,j^1,\beta^1\right)A\left(j^1,\beta^1,j^2,\beta^2\right)A\left(j^2,\beta^2,j,\beta\right)}{(\Lambda_N - \lambda_{j^1,\beta^1})(\Lambda_N - \lambda_{j^2,\beta^2})}]c(N,j,\beta)$$
$$+ \sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(\beta_3,j_3)\in Q(\rho^\alpha,6r_3)\\(j^2,\beta^2)\neq(j,\beta)\\(j^3,\beta^3)\neq(j,\beta)}} \frac{A\left(j,\beta,j^1,\beta^1\right)A\left(j^1,\beta^1,j^2,\beta^2\right)A\left(j^2,\beta^2,j^3,\beta^3\right)}{(\Lambda_N - \lambda_{j^1,\beta^1})(\Lambda_N - \lambda_{j^2,\beta^2})}c(N,j^3,\beta^3)$$
$$+ O(\rho^{-p\alpha}). \tag{63}$$

In this way, iterating $2p$ times, we get

$$(\Lambda_N - \lambda_{j,\beta})c(N,j,\beta) = \left[\sum_{k=1}^{2p} S_k^{'}\right] c(N,j,\beta) + C_{2p}^{'} + O(\rho^{-p\alpha}), \tag{64}$$

where

$$S_k^{'}(\Lambda_N, \lambda_{j,\beta}) = \sum_{\substack{(\beta_1,j_1)\in Q(\rho^{\alpha},6r_1)\\ (\beta_{k+1},j_{k+1})\in Q(\rho^{\alpha},6r_{k+1})\\ (j^{k+1},\beta^{k+1})=(j,\beta)\\ (j^s,\beta^s)\neq(j,\beta),\ s=2,\ldots,k}} \left(\prod_{i=1}^{k} \frac{A\left(j^{i-1},\beta^{i-1},j^i,\beta^i\right)}{(\Lambda_N - \lambda_{j^i,\beta^i})}\right) A\left(j^k,\beta^k,j,\beta\right) \tag{65}$$

and

$$C_k^{'} = \sum_{\substack{(\beta_1,j_1)\in Q(\rho^{\alpha},6r_1)\\ (\beta_{k+1},j_{k+1})\in Q(\rho^{\alpha},6r_{k+1})\\ (j^s,\beta^s)\neq(j,\beta),\ s=2,\ldots,k+1}} \left(\prod_{i=1}^{k} \frac{A\left(j^{i-1},\beta^{i-1},j^i,\beta^i\right)}{(\Lambda_N - \lambda_{j^i,\beta^i})}\right) A\left(j^k,\beta^k,j^{k+1},\beta^{k+1}\right) c(N,j^{k+1},\beta^{k+1}). \tag{66}$$

Now we estimate $S_k^{'}$ and $C_k^{'}$. For this, we consider the terms which appear in the denominators of (65) and (66). By the conditions under the summations in (65) and (66), we have $j_1 + j_2 + \ldots + j_i \neq 0$ or $\beta_1 + \beta_2 + \ldots + \ldots \beta_i \neq 0$, for $i = 2,3,\ldots,k$.

If $\beta_1 + \beta_2 + \ldots + \ldots \beta_i \neq 0$, then by (54) and (59), we have

$$|\Lambda_N - \lambda_{j^i,\beta^i}| > \frac{1}{2}\rho^{\alpha_2}. \tag{67}$$

If $\beta_1 + \beta_2 + \ldots + \ldots \beta_i = 0$, i.e., $j_1 + j_2 + \ldots + j_i \neq 0$, then by a well-known theorem

$$|\lambda_{j,\beta} - \lambda_{j^i,\beta^i}| = |\mu_j - \mu_{j^i}| > c_{17},$$

hence by (59), we obtain

$$|\Lambda_N - \lambda_{j^i,\beta^i}| > \frac{1}{2}c_{18}. \tag{68}$$

Since $\beta_k \neq 0$ for all $k \leq 2p$, the relation $\beta_1 + \beta_2 + \ldots + \ldots \beta_i = 0$ implies $\beta_1 + \beta_2 + \ldots + \ldots \beta_{i\pm1} \neq 0$. Therefore the number of multiplicands $\Lambda_N - \lambda_{j^i,\beta^i}$ in (66) satisfying (67) is no less then p. Thus, by (45), (67) and (68), we get

$$S_1^{'} = O(\rho^{-\alpha_2}), \quad C_{2p}^{'} = O(\rho^{-p\alpha_2}) \tag{69}$$

Theorem 3.2. *(a) For every eigenvalue $\lambda_{j,\beta}$ of $L(P(s))$ such that $\beta + j\delta \in V_{\delta}^{'}(\rho^{\alpha_1})$, there exists an eigenvalue Λ_N of the operator $L(V)$ satisfying*

$$\Lambda_N = \lambda_{j,\beta} + E_{k-1} + O(\rho^{-k\alpha_2}), \tag{70}$$

where $E_0 = 0$, $E_s = \sum_{k=1}^{2p} S_k^{'}(E_{s-1} + \lambda_{j,\beta}, \lambda_{j,\beta})$, $\quad s = 1,2,\ldots$

(b) If

$$|\Lambda_N - \lambda_{j,\beta}| < c_{19} \tag{71}$$

and

$$|c(N, j, \beta)| > \rho^{-q\alpha} \tag{72}$$

hold then Λ_N *satisfies* (70).

Proof. By Lemma 2.3 $(a) - (b)$, there exists N satisfying the conditions (71) and (72) in part (b). Hence it sufficesto prove part (b). By (54) and (71), the triples (N, j^k, β^k) satisfy the iterability condition in (49). Hence we can use (64) and (69). Now we prove the theorem by induction:

For $k = 1$, to prove (70), we divide both sides of the equation (64) by $c(N, j, \beta)$ and use the estimations (69).

Suppose that (70) holds for $k = s$, i.e.,

$$\Lambda_N = \lambda_{j,\beta} + E_{s-1} + O(\rho^{-s\alpha_2}). \tag{73}$$

To prove that (70) is true for $k = s + 1$, in (64) we substitute the expression (73) for Λ_N into $\sum_{k=1}^{2p} S_k^{'}(\Lambda_N, \lambda_{j,\beta})$, then we get

$$(\Lambda_N - \lambda_{j,\beta})c(N, j, \beta) = \left[\sum_{k=1}^{2p} S_k^{'}\left(\lambda_{j,\beta} + E_{s-1} + O(\rho^{-s\alpha_2}), \lambda_{j,\beta}\right)\right] c(N, j, \beta) + C_{2p}^{'} + O(\rho^{-p\alpha}) \tag{74}$$

dividing the both sides of the last equality by $c(N, j, \beta)$ and using Lemma 2.3-(ii), we obtain

$$\Lambda_N = \lambda_{j,\beta} + \sum_{k=1}^{2p} S_k^{'}\left(\lambda_{j,\beta} + E_{s-1} + O(\rho^{-s\alpha_2}), \lambda_{j,\beta}\right) + O(\rho^{-(p-q)\alpha}). \tag{75}$$

Now we add and subtract the term $\sum_{k=1}^{2p} S_k^{'}\left(E_{s-1} + \lambda_{j,\beta}, \lambda_{j,\beta}\right)$ in (75), then we have

$$\Lambda_N = \lambda_{j,\beta} + E_s + O(\rho^{-(p-q)\alpha}) + \left[\sum_{k=1}^{2p} S_k^{'}\left(\lambda_{j,\beta} + E_{s-1} + O(\rho^{-s\alpha_2}), \lambda_{j,\beta}\right) - \sum_{k=1}^{2p} S_k^{'}\left(E_{s-1} + \lambda_{j,\beta}, \lambda_{j,\beta}\right)\right]. \tag{76}$$

Now, we first prove that $E_j = O(\rho^{-\alpha_2})$ by induction. $E_0 = 0$. Suppose that $E_{j-1} = O(\rho^{-\alpha_2})$, then $a = \lambda_{j,\beta} + E_{j-1}$ satisfies (67) and (68). Hence we get

$$S_1^{'}(a, \lambda_{j,\beta}) = O(\rho^{-\alpha_2}) \Rightarrow E_j = O(\rho^{-\alpha_2}). \tag{77}$$

To prove the theorem, we need to show that the expression in the square brackets in (76) is equal to $O(\rho^{-(s+1)\alpha_2})$. This can be easily checked by (77) and the obvious relation

$$\frac{1}{\lambda_{j,\beta} + E_{s-1} + O(\rho^{-s\alpha_2}) - \lambda_{j^k,\beta^k}} - \frac{1}{\lambda_{j,\beta} + E_{s-1} + \lambda_{j^k,\beta^k}} = O(\rho^{-(s+1)\alpha_2}), \tag{78}$$

for $\beta^k \neq \beta$. The theorem is proved. □

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